Section 2.3 Differentiation

We learn:

- The definition and interpretation as slope of partial derivatives.
- How to compute partial derivatives.
- Linear approximations to a function
- The tangent plane of a function of two variables
- The gradient of a function

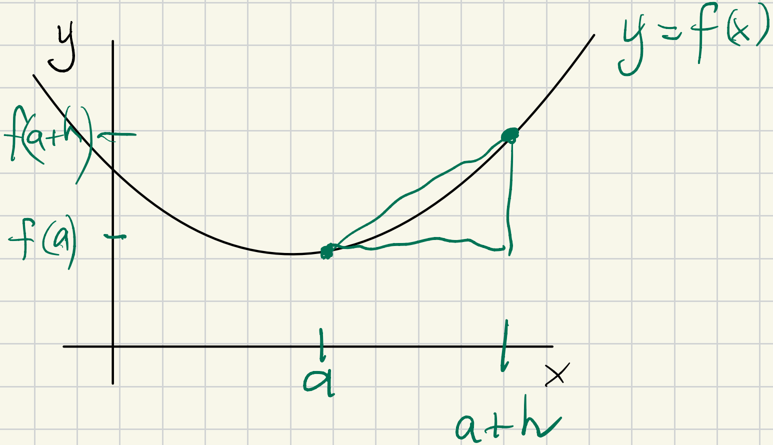
In the book there are also some theoretical things:

- what it means to be differentiable
- Theorem 9 gives a condition a function to be differentiable.

Before we get started: review of the derivative of a function $f : R \to R$.

We know the derivative of f at the point a is

$$f'(a) = \frac{df}{dx}\Big|_a = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



- It represents the rate of change: how fast we are going.
- It provides a linear approximation to $f(x)$ near $x = a$.

$$g(x) = f(a) + (x-a) f'(a)$$

has graph = the tangent line at a to the graph of f.

Definition of partial derivatives

To make things easier we start with a function f : R^n -> R
The (partial) derivative of f(x_1, … ,x_n) with respect to variable x_j at the point a = (a_1, … , a_n) is

$$\left.\frac{\partial f}{\partial x_j}\right|_{\underline{x}=\underline{a}} = \lim_{h\to 0} \frac{f(a_1,\dots,a_j+h,\dots a_n) - f(a_1,\dots,a_n)}{h}$$

$$= f_{x_j}(\underline{a})$$

Idea: we regard all variables other than x_j as constants and differentiate as usual with respect to x_j.

Examples:

a. f(x,y) = 2x - y To get $\frac{\partial f}{\partial x}$

we regard y as constant:

$$\frac{\partial f}{\partial x} = 2 \qquad \frac{\partial f}{\partial y} = -1$$

The graph of f is the plane in R^3 that is the set of points (x, y, 2x-y). It is given by the equation z = 2x-y.
The partial derivatives are the slopes of this plane in the x and in the y directions.

b. The partial derivatives of x^3y + xy^2 at the point (1,2).

$$\frac{\partial (x^3y+xy^2)}{\partial x} = 3x^2y + y^2$$

Pre-class Warm-up !!

Let $f(x,y) = 5xy^4 + x^2 \sin(y) + y$
What is $\partial f/\partial x$?

a. $5y^4 + 2x\sin(y)$ ✓

b. $5y^4 + 2x\sin(y) + 1$

c. $20y^3 + 2x\cos(y) + 1$

d. The question doesn't mean anything, because we only know how to find the partial derivative of f at a point, and no point is given.

e. None of the above.

I think (hope) the Canvas site is now working!?
If you have issues, please let me know in as much precision as possible.

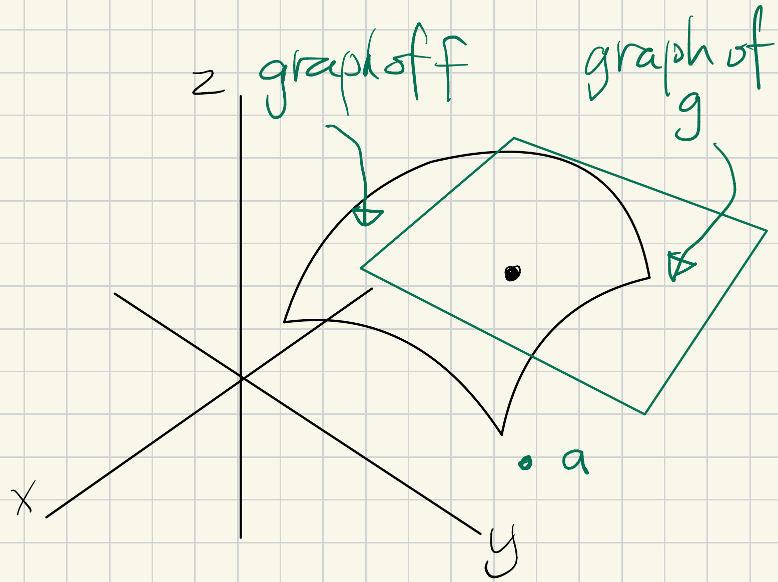
Linear approximation to f

$a = (a_1, \dots, a_n)$

g(x_1, … ,x_n) = f(a_1, … , a_n) +

$$\frac{\partial f}{\partial x_1}(a)(x_1 - a_1) + \frac{\partial f}{\partial x_2}(a)(x_2 - a_2) + \cdots + \frac{\partial f}{\partial x_n}(a)(x_n - a_n)$$

The graph of the linear approximation is a linear space tangent to the graph of f at the point a.



The matrix of partial derivatives

Component
Coordinate functions:

So far we did functions f : R^n -> R.
Now we do f : R^n -> R^m $\quad f : \mathbb{R}^2 \to \mathbb{R}^3$
Example f(s,t) = s(1,2,3) + t^2(1,0,-1).
This f is made up of 3 functions R^2 -> R

$$f(s,t) = (s+t^2, 2s, 3s-t^2)$$
$$= (f_1(s,t), f_2(s,t), f_3(s,t))$$

where $f_1(s,t) = s+t^2$, $f_2(s,t) = 2s$
$f_3(s,t) = 3s - t^2$. We can do

$$\frac{\partial f_2}{\partial t} = 0 \qquad \frac{\partial f_2}{\partial s} = 2$$

We make a matrix of partial derivatives where the (i,j) entry is ∂f_i / ∂x_j

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & & \end{bmatrix} = D$$

Ex.
$$\begin{bmatrix} \frac{\partial f_1}{\partial s} = 1 & 2t \\ \frac{\partial f_2}{\partial s}\ 2 & 0 \\ 3 & -2t \end{bmatrix}$$

Is this matrix 2 x 3 or 3 x 2 ✓

This matrix is called the derivative (matrix) of f , or the Jacobian matrix of f.

Definition of differentiability

Functions might not be differentiable everywhere. The official definition for $f : R^n \to R^m$ to be differentiable at a point a in R^n is as follows.

Let $D(a)$ be the derivative matrix of f at a. Then f is differentiable at a if $D(a)$ exists and also

matrix · vector

$$\lim_{x \to a} \frac{\| f(x) - f(a) - Df(a) \cdot (x-a) \|}{\| x - a \|} = 0$$

The function
$g(x) = f(a) + Df(a)\,(x-a)$
is the best linear approximation to f near a.

Theorem 8: If f is differentiable at a then f is continuous at a.

Theorem 9: If all the partial derivatives $\partial f_i / \partial x_j$ exist and are continuous near a then f is differentiable at a.

In the book they also define the gradient of a function $f : R^n \to R$ but do not explain why. Their notation confuses row vectors and the column vectors used in matrix multiplication.

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The gradient of f at a is the vector

$$\begin{bmatrix} \frac{\partial f}{\partial x_1}(a) \\ \vdots \\ \frac{\partial f}{\partial x_n}(a) \end{bmatrix}$$

Example. Find the gradient of
f(x,y) = x^2y - xy^2
at the point (1,-1)

$z = 2x - y + 5$

What is the slope of the graph of f(x,y) = 2x - y + 5 in the direction of increasing x at the point (1,3)?

a. 1

b. 2 ✓ $= \frac{\partial f}{\partial x}(1,3)$

c. 3

Example. Let f(x,y) = 3xy^2 + x^3 + 1

a. Find the equation of the tangent plane to the graph of f at (x,y) = (1,-1).

b. Use the linear approximation of f around (x,y) = (1,-1) to approximate f(0.9,-1.1).

a. It is $z = \frac{\partial f}{\partial x}(1,-1)x + \underbrace{\frac{\partial f}{\partial y}(1,-1)y} + D$

$= 6x - 6y + D$

Plug in $z = f(1,-1) = 5$ $x = 1$ $y = -1$

$5 = 6 + 6 + D$ $D = -7$

$z = 6x - 6y - 7$

b. $g(x,y) = 6x - 6y - 7$

Find $g(0.9, -1.1)$.